Flexural Vibrations at Right Angles to the Plane of the Ring

Assuming that the angle of twist is very small, the vibrations are governed approximately by

$$(\partial^4 u/\partial\phi^4) + (\rho A a^4/EI_\phi)(\partial^2 u/\partial t^2) = 0 \qquad (A.26)$$

where u is the displacement at right angles to the ring and I_ϕ the moment of inertia of the section about an axis in the plane of the ring. Equation (A26) is identical to that of the vibrations of a straight beam, and Eqs. (5) and (6) may be readily deduced. The only difference between the straight beam and the ring is that the shearing force is a function of the torsional stiffness of the ring, and torsional moments are induced (for further details see Love[2]). Strictly speaking, the uncoupling of the out of plane vibrations and torsional vibrations is not possible. Since the gridwork model can, in any event, only approximately represent motion involving primarily u displacements, the simplified equation given is felt to be sufficiently accurate, especially for small arc length/radius ratios.

Lumped Mass

When the ring is supposed massless, its deflection is governed by

$$(d^6V/d\phi^6) + 2(d^4V/d\phi^4) + (d^2V/d\phi^2) = 0 \qquad (A27)$$

together with the condition of inextensibility $W = dV/d\phi$. The general solution of (A27) is

$$V = B_1 + B_2\phi + B_3\cos\phi + B_4\sin\phi + B_5\phi\cos\phi + B_6\phi\sin\phi \qquad (A28)$$

One may now proceed as in the distributed mass formulation and obtain the relation (A17). The matrix (P_{ir}) in this case can be evaluated explicitly without great difficulty. Writing Eqs. (A16) in the form of Eq. (1), the c_i are given by Eqs. (26).

References

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Free Vibrations of Ring-Stiffened Conical Shells

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Linear shell theory was used to develop an analysis for predicting the natural frequencies of ring-stiffened simply supported conical shells. An experimental investigation was performed in conjunction with this analysis. In the analysis, the longitudinal, circumferential, and rotatory inertia forces were assumed to be small in comparison with the radial inertia force. The previous assumption simplified the uncoupling of the equilibrium equations. A linear Donnell-type vibration equation was obtained for orthotropic conical shells. By finding an equivalent orthotropic shell, the free vibration characteristics of a ring-stiffened conical shell were determined. Application of the Galerkin method reduced the shell equations to the form of frequency determinant. Digital computer techniques were used to solve the resulting frequency determinant.

Nomenclature

A_n	= displacement coefficient, see Eqs. (5)
A_s	= stiffener area
B_n	= displacement coefficient, see Eqs. (5)
C_n	= displacement coefficient, see Eqs. (5)
D	= flexural stiffness of shell wall $[Eh^3/12(1 - \nu^2)]$
E	= Young's modulus for isotropic material
E_x, E_ϕ	= Young's modulus for orthotropic material
$F_1(n)$–$F_5(n)$	= expressions defined by Eqs. (14–18)
f	= frequency (cps) ($f = \omega/2\pi$)
$G_1(n, m)$, $G_2(n, m)$, $G_3(n, m)$	= expressions defined by Eqs. (11–13)
G	= shear modulus
h	= thickness of the shell
h_ϕ	= effective thickness in circumferential direction
I_ϕ	= effective moment of inertia of ring and shell combination, see Eq. (21)
I	= imaginary part of the expression, see Eqs. (5)
I_0	= moment of inertia of the stiffener about its own centroid
i	= $(-1)^{1/2}$
k_1	= see Eq. (22)
k_2	= see Eq. (23)
K^4	= geometry parameter $[12\mu(x_1/h)^2]$
l	= slant length
l_s	= length between stiffeners
L_1–L_8	= differential operators
L_7^{-1}	= inverse operator
m, n	= integers
R_1, R_2	= radii of small and large end of truncated cone, respectively
s	= complex number $[\gamma + in\beta]$
t	= cone circumferential wave number
t	= time

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u, v, w = nondimensionalized displacements, with respect to x_1, of a point on the middle surface of the shell in the meridional, circumferential, and normal directions, respectively
x = coordinate in the meridional direction
x_1 = distance of the top of a truncated cone from the vertex
x_2 = distance of the bottom of a truncated cone from the vertex nondimensionalized with respect to x_1
z = coordinate in the normal direction
z_c = distance of the over-all centroid of the ring shell combination from the middle surface (Fig. 2)
α = semivertex angle of conical shell
β = see Eq. (6)
γ = see Eq. (6)
δ_1 = see Eqs. (19)
δ_2 = see Eqs. (19)
η_1, η_2, η_3 = extensional and shearing rigidities of an orthotropic shell $|hE_x/\mu, hE_\phi/\mu, Gh|$
μ = Poisson ration parameter $|1 - \nu_{x\phi}\nu_{\phi x}|$
$\nu_{x\phi}, \nu_{\phi x}$ = Poisson's ratios for orthotropic shell
ρ = density of the shell material
ϕ = angle denoting the circumferential location of a point on the shell middle surface
Ω = frequency parameter, see Eq. (10) $|[12(x_1/h)^2 (x_1{}^2\rho\mu/E_x)\omega^2]^{1/2}|$
$\bar{\Omega}$ = frequency parameter, see Table 1
ω = circular frequency $(2\pi f)$
∇^2 = $(\partial^2/\partial x^2) + (1/a^2)(\partial^2/\partial\phi^2)$

Subscripts following a comma denote differentation.

Introduction

Although the problem of determining the natural frequencies of isotropic conical shells has produced many papers[1-19] the literature reveals only two investigations on the free vibrations of orthotropic conical shells. Kutnikova and Sakharov's[20] orthotropic analysis was limited to the calculation of the fundamental vibration mode, and the analysis of Godzevich[21] was limited to a one-term radial mode approximation. It is apparent that no analysis is available for accurately predicting the complete frequency spectra of the ring-stiffened simply supported conical shells. In addition, the author could not find any experimental data on this problem.

This work attempts to fill these voids. The analysis is based on the linear Donnell-type equations for orthotropic conical shells derived by Singer.[22,23] These equations are modified to include dynamic terms for the free vibration case. The method given by Bodner[24] for reducing a ring-stiffened cylindrical shell to an orthotropic cylindrical shell is used in the development of the conical shell equations. By finding an equivalent orthotropic shell the free vibration solution can

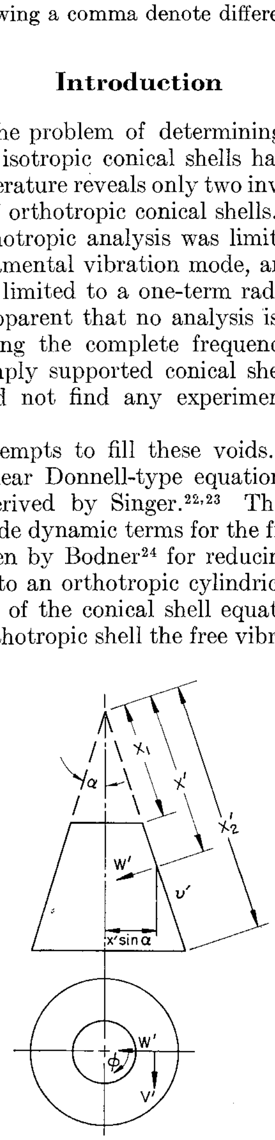

Fig. 1 Displacement and coordinate notation.

be applied to ring-stiffened conical shells. The orthotropic shell equations are solved by the Galerkin method, resulting in an infinite frequency determinant. This determinant is truncated and solved by matrix iteration techniques for the eigenvalues that are the resonant frequencies. The size of the determinant was increased until convergence was obtained on the eigenvalues. The forementioned calculations necessitated the development of a program for use on a digital computer.

An experimental investigation was performed in conjunction with this analysis. The frequency spectra of isotropic and ring-stiffened conical shells supported at each end were obtained and compared with the developed theory. Values obtained from this analysis were in good agreement with the experimental data.

Analysis

The present investigation considers vibrations in which there is both stretching and bending of the shell. The modal shapes associated with this type of vibration are called "breathing" or "lobar" modes. Calculation of the natural frequencies of these modal shapes will now be considered.

If the longitudinal and circumferential inertia terms are neglected, then the nondimensional Donnell-type uncoupled equations of motion[22] for an orthotropic conical shell (Fig. 1) are given by

$$L_5(u) = \cot\alpha\{L_4[x\nu_{\varphi x}w_{,x} - (\eta_2/\eta_1)w] - (\eta_2/\eta_3)(1/\sin\alpha)L_2[w_{,\varphi}]\} \quad (1)$$

$$L_6(v) = \cot\alpha\{L_3[x\nu_{\varphi x}w_{,x} - (\eta_2/\eta_1)w] - (\eta_2/\eta_3)(1/\sin\alpha)L_1[w_{,\varphi}]\} \quad (2)$$

$$(1/x)L_8(w) + (1/x^3)\eta_2/\eta_1 K^4 \cot^2\alpha L_7{}^{-1} \times [(1/x)(x^3 w_{,xx})_{,xx}] + 12(x_1/h)^2(x_1{}^2\rho\mu/E_x)w_{,tt} = 0 \quad (3)$$

where

$$L_1(\ ) = x^2(\ )_{,xx} + x(\ )_{,x} + (\eta_3/\eta_1)(1/\sin^2\alpha)(\ )_{,\varphi\varphi} - \mathrm{n}_2/\eta_1(\ )$$

$$L_2(\ ) = (\eta_2/\eta_1)(1/\sin\alpha)\{[\nu_{x\varphi} + (\eta_3/\eta_2)]\, x(\ )_{,x\varphi} - [1 + (\eta_3/\eta_2)](\ )_{,\varphi}\}$$

$$L_3(\ ) = (\eta_2/\eta_3)(1/\sin\alpha)\{[\nu_{x\varphi} + (\eta_3/\eta_2)]\, x(\ )_{,x\varphi} + [1 + (\eta_3/\eta_2)](\ )_{,\varphi}\}$$

$$L_4(\ ) = x^2(\ )_{,xx} + x(\ )_{,x} + (\eta_2/\eta_3)(1/\sin^2\alpha)(\ )_{,\varphi\varphi} - (\ )$$

$$L_5(\ ) = L_4[L_1(\ )] - L_2[L_3(\ )]$$

$$L_6(\ ) = L_3[L_2(\ )] - L_1[L_4(\ )]$$

$$L_7(\ ) = x(\ )_{,xxxx} + 2(\ )_{,xxx} + (\eta_2/\eta_1)[-(\ )_{,xx}/x + (\ )_{,x}/x^2 + 2(\ )_{,\varphi\varphi}/x^3\sin^2\alpha + (\ )_{,\varphi\varphi\varphi\varphi}/x^3\sin^4\alpha] + [\mu/2(\eta_2/\eta_3) - \nu_{\varphi x}](2/\sin^2\alpha)[(\ )_{,xx\varphi\varphi}/x - (\ )_{,x\varphi\varphi}/x^2 + (\ )_{,\varphi\varphi}/x^3]$$

$$L_7 L_7{}^{-1}(\ ) = (\ )$$

$$L_8(\ ) = x(\ )_{,xxxx} + 2(\ )_{,xxx} + (\eta_2/\eta_1)[-(\ )_{,xx}/x + (\ )_{,x}/x^2 + 2(\ )_{,\varphi\varphi}/x^3\sin^2\alpha + (\ )_{,\varphi\varphi\varphi\varphi}/x^3\sin^4\alpha] + [2(\eta_3/\eta_1) + \nu_{\varphi x}] \times (2/\sin^2\alpha)[(\ )_{,xx\varphi\varphi}/x - (\ )_{,x\varphi\varphi}/x^2 + (\ )_{,\varphi\varphi}/x^3]$$

The simple-support condition that will be considered approximates the conventional simple-support condition. This condition is given by

$$\begin{aligned} w &= 0 \qquad \text{at } x = 1 \text{ and } x = x_2 \\ w_{,xx} + \nu_{\phi x}(w_{,x}/x) &= 0 \text{ at } x = 1 \text{ and } x = x_2 \end{aligned} \quad (4)$$

The usual simple-support requirements of $v = 0$ and $u \neq 0$

are not satisfied. Instead, the displacements in the u and v directions are resisted by elastic supports. Singer,[25] using the forementioned approximate simple-support boundary conditions, has shown for the buckling problem that these conditions give the same numerical results as obtained by Seide[26] using the more restrictive boundary conditions.

The displacements that will be used in the solution are the same as those used by Singer[25] and Hoff and Singer.[27] These are of the form

$$\left.\begin{aligned} u &= I \sum_{n=1}^{\infty} A_n x^s \sin t\,\varphi \cos\omega t \\ v &= I \sum_{n=1}^{\infty} B_n x^s \cos t\,\varphi \cos\omega t \\ w &= I \sum_{n=1}^{\infty} C_n x^s \sin t\,\varphi \cos\omega t \end{aligned}\right\} \tag{5}$$

where I indicates the imaginary part of the expressions, A_n and B_n are complex coefficients, and C_n and t are real. The number of circumferential waves is denoted by t, and s is the complex number

$$s = \gamma + in\beta \tag{6}$$

γ and β are real and are evaluated from the boundary conditions, n is an integer, and $i = [-1]^{1/2}$.

The displacements given in Eqs. (5) can only be made to satisfy the equilibrium equations, Eqs. (1) and (2). The remaining equilibrium equation, Eq. (3), will be solved approximately by the Galerkin method.

A substitution of the displacement functions of Eqs. (5) into Eqs. (1) and (2) yields

efficients of Eqs. (10) and finding the eigenvalues Ω^2 by matrix iteration for each value of t. The numerical accuracy of the eigenvalues can be obtained to any degree by including more terms in the truncated determinant.

The expressions for G_1, G_2, and G_3† in Eqs. (10) are

$$G_1(n, m) = F_1(n)\left[\frac{m+n}{4(\gamma-1)^2+(m+n)^2\beta^2} + \frac{m-n}{4(\gamma-1)^2+(m-n)^2\beta^2}\right] + F_2(n)\left[\frac{1}{4(\gamma-1)^2+(m+n)^2\beta^2} - \frac{1}{4(\gamma-1)^2+(m-n)^2\beta^2}\right] \tag{11}$$

$$G_2(n, m) = F_3(n)\left[\frac{m+n}{4\gamma^2+(m+n)^2\beta^2} + \frac{m-n}{4\gamma^2+(m-n)^2\beta^2}\right] + F_4(n)\left[\frac{1}{4\gamma^2+(m+n)^2\beta^2} - \frac{1}{4\gamma^2+(m-n)^2\beta^2}\right] \tag{12}$$

$$G_3(n, m) = 2(\gamma+1)\left[\frac{1}{4(\gamma+1)^2+(m+n)^2\beta^2} - \frac{1}{4(\gamma+1)^2+(m-n)^2\beta^2}\right] \tag{13}$$

$$F_1(n) = 2n\beta^2(\gamma-1)[(\gamma-1)^2+\gamma(\gamma-2) - (\eta_2/\eta_1) - 2n^2\beta^2 - 2\delta_1(t^2/\sin^2\alpha)] \tag{14}$$

$$A_n = \frac{\cos\alpha\,\sin\alpha\left(\dfrac{\eta_2}{\eta_3}\right)t^2\left[\left(\dfrac{\eta_2}{\eta_1}\nu_{x\varphi} - \nu_{\varphi x} + \dfrac{\eta_3}{\eta_1}\right)s - \dfrac{\eta_3}{\eta_1}\right] + \left(\nu_{\varphi x}s - \dfrac{\eta_2}{\eta_1}\right)\dfrac{\eta_3}{\eta_1}\,(s^2-1)\sin^2\alpha}{t^4\dfrac{\eta_2}{\eta_1} + t^2\sin^2\alpha\left[\left(\dfrac{\eta_2}{\eta_1}\dfrac{\eta_2}{\eta_3}\,\nu^2x\varphi + 2\dfrac{\eta_2}{\eta_1}\nu_{x\varphi} - \dfrac{\eta_2}{\eta_3}\right)s^2 - 2\dfrac{\eta_2}{\eta_1}\right] + \left(s^2 - \dfrac{\eta_2}{\eta_1}\right)(s^2-1)\sin^4\alpha}\,C_n \tag{7}$$

$$B_n = \frac{\cos\alpha\,\dfrac{\eta_2}{\eta_3}\,t\left[-t^2\dfrac{\eta_3}{\eta_1} + \sin^2\alpha\,1 - \nu_{\varphi x}\left(\nu_{x\varphi} + \dfrac{\eta_3}{\eta_2}\right)\right]s^2 + \left[\dfrac{\eta_2}{\eta_1}\left(\nu_{x\varphi} + \dfrac{\eta_3}{\eta_2}\right) - \left(\dfrac{\eta_3}{\eta_2} + 1\right)\nu_{\varphi x}\right]s + \dfrac{\eta_3}{\eta_1}}{t^4\dfrac{\eta_2}{\eta_1} + t^2\sin^2\alpha\left[\left(\dfrac{\eta_2}{\eta_1}\dfrac{\eta_2}{\eta_3}\,\nu^2{}_x\varphi + 2\dfrac{\eta_2}{\eta_1}\nu_{x\varphi} - \dfrac{\eta_2}{\eta_3}\right)s^2 - 2\dfrac{\eta_2}{\eta_1}\right] + \left(s^2 - \dfrac{\eta_2}{\eta_1}\right)(s^2-1)\sin^4\alpha\,2\dfrac{\eta_2}{\eta_1}}\,C_n \tag{8}$$

that can be substituted into Eqs. (5) for determining displacements u and v once the coefficients C_n are known quantities.

The parameters β and γ are determined as

$$\beta = \pi/\ln x_2 \qquad \gamma = (1 - \nu_{\varphi x})/2 \tag{9}$$

by satisfying the boundary conditions given in Eqs. (4).

Since the exact solution of the remaining equilibrium equation, Eq. (3), is impractical to obtain, one must resort to approximate methods. The Galerkin method will be used to obtain the frequency spectrum from Eq. (3). Substitution of w from Eqs. (5) into Eq. (3) and use of the Galerkin method yields the following frequency determinant:

$$\sum_{n=1}^{\infty} C_n\{[(-1)^{m+n}x_2{}^{2\gamma-2} - 1]G_1(n,m) + K^4(\eta_2/\eta_1)\cos^2\alpha \times [(-1)^{m+n}x_2{}^{2\gamma} - 1]G_2(n,m) - \Omega^2[(-1)^{m+n}x_2{}^{2\gamma+2} - 1] \times G_3(n,m)\} = 0 \qquad m = 1, 2, 3, \ldots \tag{10}$$

where

$$\Omega^2 = 12\,(x_1/h)^2(x_1{}^2\rho\mu/E_x)\omega^2$$

The frequencies are determined by the condition that the determinant of the coefficients of Eqs. (10) must vanish for nontrival displacements. Numerical values can be obtained by the usual method of truncating the determinant of the co-

$$F_2(n) = 2(\gamma-1)\{\gamma(\gamma-1)^2(\gamma-2) + (\eta_2/\eta_1)\gamma(2-\gamma) - n^2\beta^2[\gamma(\gamma-2) + 5(\gamma-1)^2 - (\eta_2/\eta_1)] + n^4\beta^4 - 2(t^2/\sin^2\alpha) \times [(\eta_2/\eta_1) + \delta_1(\gamma-1)^2 - \delta_1 n^2\beta^2] + (\eta_2/\eta_1)(t^4/\sin^4\alpha)\} \tag{15}$$

$$F_3(n) = [2\gamma n\beta^2/F_5(n)]\Big[\sin^2\alpha\Big([(2\gamma^2-1) - 2n^2\beta^2]\{[\gamma^2 - (\eta_2/\eta_1)](\gamma^2-1) - n^2\beta^2[(6\gamma^2-1) - (\eta_2/\eta_1)] + n^4\beta^4\} - [(\gamma^4-\gamma^2) + n^2\beta^2(1-6\gamma^2) + n^4\beta^4][(2\gamma^2-1) - (\eta_2/\eta_1) - 2n^2\beta^2]\Big) - 2t^2\{[(2\gamma^2-1)\,2n^2\beta^2][\delta_2(\gamma^2-n^2\beta^2) + (\eta_2/\eta_1)] - \delta_2[(\gamma^4-\gamma^2) + n^2\beta^2(1-6\gamma^2) + n^4\beta^4]\} + (\eta_2/\eta_1)(t^4/\sin^2\alpha)[(2\gamma^2-1) - 2n^2\beta^2]\Big] \tag{16}$$

$$F_4(n) = [2\gamma/F_5(n)]\Big[\sin^2\alpha\Big([(\gamma^4-\gamma^2) + n^2\beta^2(1-6\gamma^2) + n^4\beta^4]\{[\gamma^2 - (\eta_2/\eta_1)](\gamma^2-1) - n^2\beta^2[(6\gamma^2-1) - (\eta_2/\eta_1)] + n^4\beta^4\} + 4\gamma^2n^2\beta^2[(2\gamma^2-1) - 2n^2\beta^2][(2\gamma^2-1) - (\eta_2/\eta_1) - 2n^2\beta^2]\Big) - 2t^2\{[(\gamma^4-\gamma^2) + n^2\beta^2(1-6\gamma^2) + n^4\beta^4][\delta_2(\gamma^2-n^2\beta^2) + (\eta_2/\eta_1)] + 4\delta_2\gamma^2n^2[(2\gamma^2-1) - 2n^2\beta^2]\} + (\eta_2/\eta_1)(t^4/\sin^2\alpha)[(\gamma^4-\gamma^2) + n^2\beta^2(1-6\gamma^2) + n^4\beta^4]\Big] \tag{17}$$

† The notation used herein is similar to that used by Singer.[23]

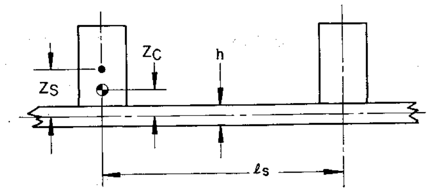

Fig. 2 Geometry of stiffeners.

$$
\begin{aligned}
F_5(n) = {} & \sin^4\alpha(\{(\gamma^2 - 1)[\gamma^2 - (\eta_2/\eta_1)] - \\
& n^2\beta^2[(6\gamma^2 - 1) - (\eta_2/\eta_1)] + n^4\beta^4\}^2 + \\
& 4\gamma^2 n^2\beta^2[(2\gamma^2 - 1) - (\eta_2/\eta_1) - 2n^2\beta^2]^2) - \\
& 4t^2 \sin^2\omega(\{(\gamma^2 - 1)[\gamma^2 - (\eta_2/\eta_1)] - \\
& n^2\beta^2[(6\gamma^2 - 1) - (\eta_2/\eta_1)] + n^4\beta^4\}[\delta_2(\gamma^2 - \\
& n^2\beta^2) + (\eta_2/\eta_1)] + 4\gamma^2 n^2\beta^2\delta_2[(2\gamma^2 - 1) - \\
& (\eta_2/\eta_1) - 2n^2\beta^2]) + t^4(4[\delta_2(\gamma^2 - n^2\beta^2) + \\
& (\eta_2/\eta_1)]^2 + 2(\eta_2/\eta_1)\{(\gamma^2 - 1)[\gamma^2 - (\eta_2/\eta_1)]\} - \\
& n^2\beta^2[(6\gamma^2 - 1) - (\eta_2/\eta_1) + n^4\beta^4] + \\
& 16\gamma^2 n^2\beta^2\delta_2^2) - 4(t^6/\sin^2\alpha)(\eta_2/\eta_1)[\delta_2(\gamma^2 - \\
& n^2\beta^2) + (\eta_2/\eta_1)] + (t^8/\sin^4\alpha)(\eta_2/\eta_1)^2 \quad (18)
\end{aligned}
$$

The method of replacing a cylindrical shell with closely spaced and identical ring stiffeners by an equivalent orthotropic cylindrical shell is not new. This method has been used by Bodner[24] for stability problems and by Hoppmann[28] for vibration problems. The method of Hoppmann,[28] however, requires three experiments for establishing equivalence, whereas the method proposed by Bodner[24] only requires computations. Bodner's equivalent orthotropic shell method has also been used by Singer[23] for stability problems of conical shells. In the present investigation, Bodner's method will be used to predict the frequency spectrum of ring-stiffened conical shells.

A diagram showing the geometry of a section of the ring-stiffened shell is given in Fig. 2. The prime assumption of Bodner's method is that the ring stiffeners have very little effect on the axial extensional and bending rigidities and the shear rigidity of the isotropic shell. These assumptions yield the following equivalent orthotropic shell properties:

$$
\left.
\begin{aligned}
&\mu = 1 - \nu^2 \qquad \eta_1 = Eh/(1 - \nu^2) \\
&\eta_2 = Eh_\varphi/(1 - \nu^2) \\
&\eta_3 = Eh/2(1 + \nu) \qquad \delta_1 = 1 \\
&\delta_2 = (1 + \nu)k_2 - \nu = \bar{\delta}_2 \qquad \gamma = (1 - \nu)/2
\end{aligned}
\right\} \quad (19)
$$

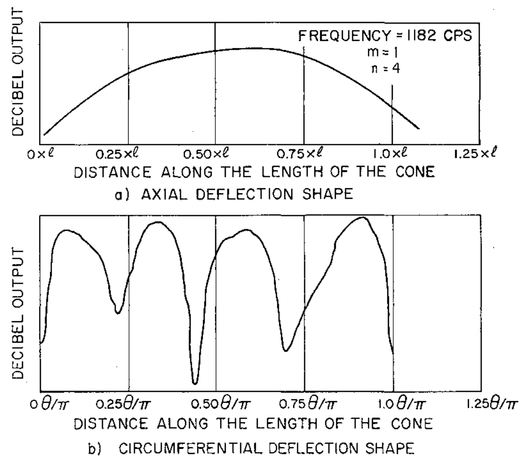

Fig. 3 A typical mode shape record.

The extensional rigidity is assumed to be proportional to the average thickness in the circumferential direction, and the bending rigidity is assumed to be proportional to the effective moment of inertia of the ring-shell combination. The circumferential equivalent thickness and moment of inertia then become

$$h_\varphi = h + A_s/l_s = k_1 h \quad (20)$$

and

$$I_\varphi = I_0 + A_s(z_s - z_c)^2 + [l_s h/(1 - \nu^2)][h^2/12 + z_c^2] \quad (21)$$

where $k_1 = 1 + (A_s/l_s h)$, A_s is the stiffener area, and I_0 is the stiffener moment of inertia about its own centroid. The geometric parameters used in Eqs. (20) and (21) are shown in Fig. 2.

The ring-stiffened shell can be reduced to an orthotropic shell by replacing the quantity η_2/η_1 in the orthotropic shell formulas with an equivalent parameter for the ring-stiffened shell. This parameter has two values depending upon whether the term η_2/η_1 refers to extensional stiffness or bending stiffness. If the term refers to extensional stiffness then

$$\eta_2/\eta_1 = h_\phi/h = k_1 \quad (22)$$

and for bending stiffness

$$\eta_2/\eta_1 = [12\ (1 - \nu^2)I_\varphi/l_s h^3] = k_2 \quad (23)$$

Equations (10) were programed on an IBM 7090 digital computer. The matrix iteration technique was used to obtain the three lowest eigenfrequency parameters and accompanying eigenvectors of the resulting dynamical matrix.

Table 1 Frequency parameter comparison of orthotropic conical shells with equivalent orthotropic cylindrical shells[a]

		Conical shell			Cylindrical shell[b]		
E_ϕ/E_x	t	1	2	3	1	2	3
0.02	0	29.18	38.05	48.15	43.46	43.78	45.37
	3	7.26	18.83	28.35	7.77	21.22	31.40
	6	1.88	7.61	15.56	2.02	8.38	17.01
	9	1.61	5.07	10.85	1.41	4.86	11.00
	12	2.76	6.42	11.78	2.57	4.87	9.71
	15	4.95	10.07	17.10	5.60	7.47	11.76
	18	8.59	15.97	25.84	11.25	13.04	17.21
50	0	26,540.51	53,454.28	59,419.53	108,679.74	108,572.32	108,564.04
	3	108.64	849.98	2,563.52	121.24	1,020.16	3,087.68
	6	303.64	648.28	1,102.63	344.72	442.34	746.68
	9	996.97	1,543.22	2,144.33	1,705.07	1,736.59	1,833.89
	12	2,633.27	3,520.09	4,415.69	5,384.51	5,402.42	5,464.43
	15	5,855.81	7,311.60	8,726.67	13,143.70	13,156.38	13,217.88

[a] Frequency parameter: $\bar{\Omega}^2 = (12\ \mu\rho\omega^2/h^2E_x)$ Conical shell properties: small radius = 2.13, slant length = 8, semivertex angle = 20°, thickness = 0.02, $\nu_{\phi x}$ − 0.0. Cylindrical shell properties: radius = 3.73, length = 8, thickness = 0.02.
[b] In this table, $t = n$, not $n/\cos\alpha$.

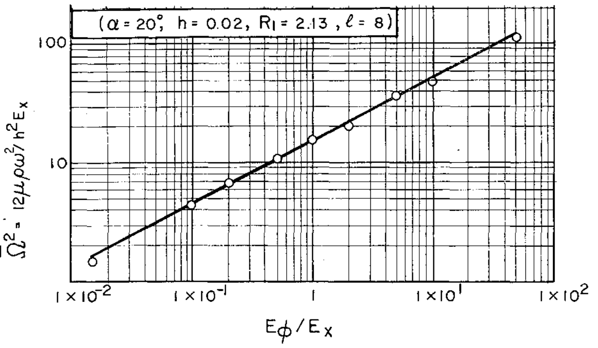

Fig. 4 Effect of circumferential stiffness on the minimum value of the frequency parameter of a conical shell.

The size of the dynamical matrix was increased until the three lowest eigenfrequency parameters converged. Convergence usually occurred with matrices of twentieth or smaller order. Axial modal shapes for a particular eigenfrequency were obtained by substituting the associated eigenvectors into the radial deflection series given in Eqs. (5) for a given geometry. The forementioned procedure is repeated for all desired circumferential wave numbers t.

Experimental Investigation

An electromagnet was used to excite the specimen. The electromagnet was fabricated by winding thin insulated copper wire over a laminated soft iron core. The magnet was designed so that it could be placed near the test specimen and induce vibrations of the cylinder walls by alternating magnetic field.

The resonance points and associated modal shapes of the vibrating shell were obtained by a method similar to that used by Gottenberg.[29] A microphone placed on a track was positioned normal to the test specimen. The microphone could be moved along the length of the specimen and also could be rotated around the specimen. It was found necessary to rotate the microphone rather than the shell, since the modal pattern stayed fixed with respect to the electromagnet. The geometric position of the microphone was determined by using a balance circuit. The voltage output of the circuit was proportional to the position of the microphone. The output of this circuit was recorded on the x axis of an x-y plotter. The microphone output was recorded on the y axis.

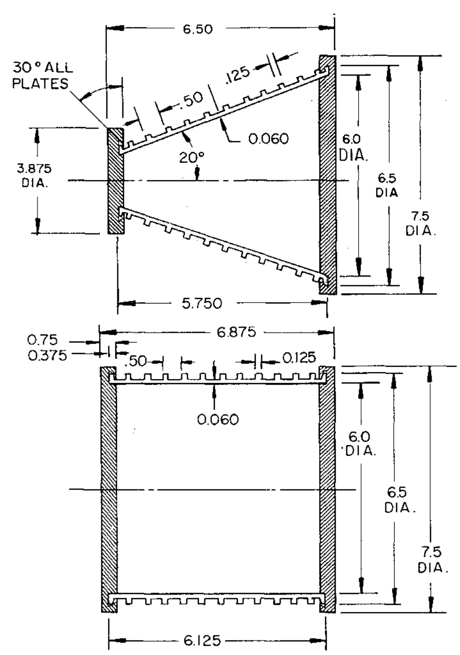

Fig. 5 Ring-stiffened conical and cylindrical shell geometry.

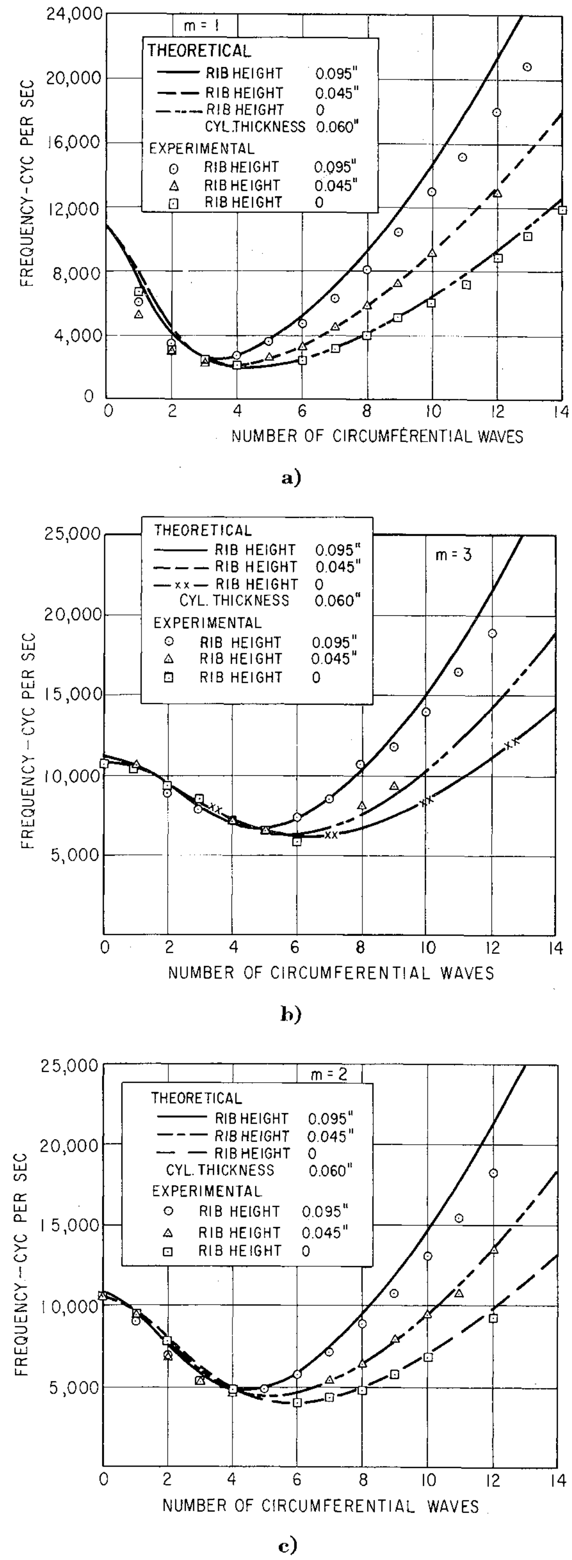

Fig. 6 Graphical comparison of experimental results with theory (ring-stiffened cylinder).

The specimens were held on centers and prevented from rotating by a friction screw. The test procedure consisted of varying the electromagnet's excitation frequency by means of an oscillator until a resonant frequency was reached. This frequency was accurately measured by means of a counter. The microphone was moved first in the axial and then in the circumferential direction around the specimen. A typical

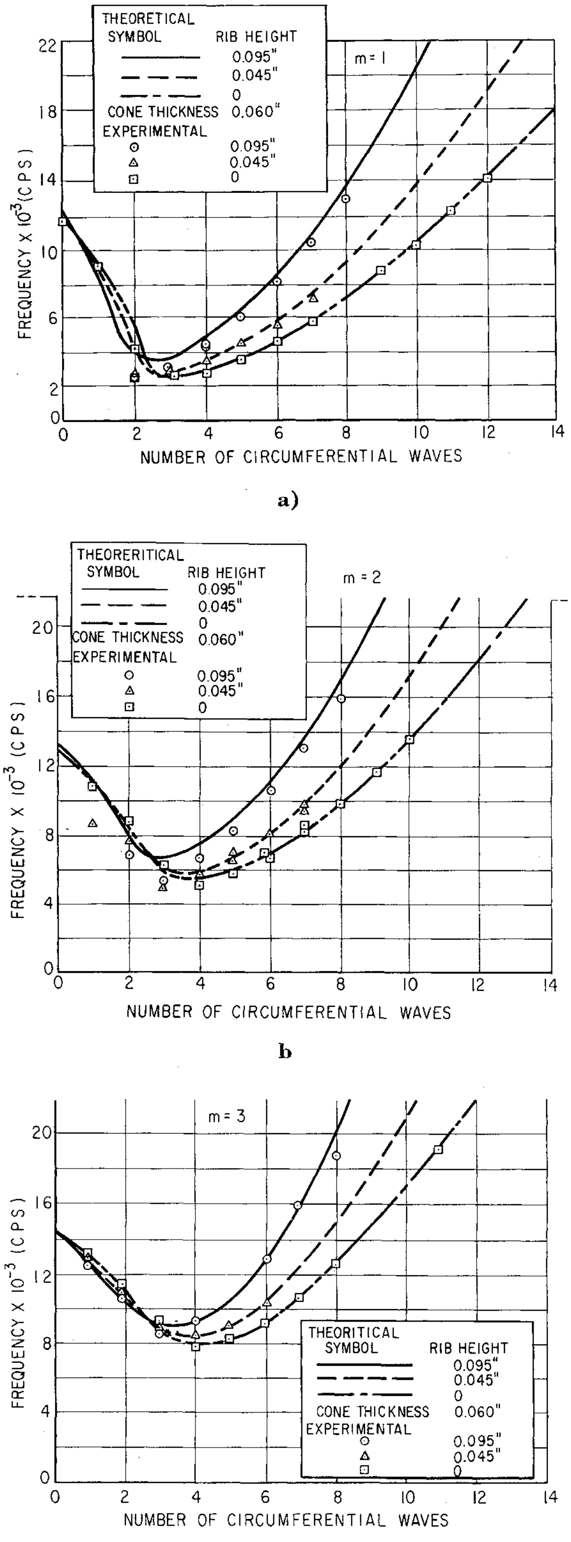

Fig. 7 Comparison of experimental results with theory (ring-stiffened conical shells).

experimental record appears in Fig. 3. The minimum points appearing in this figure are nodes, since the voltage output is approximately proportional to the displacement.

Aluminum was used for manufacturing the ring-stiffened specimens. The conical shell and cylindrical shell specimens were machined from a solid bar. The ring stiffeners initially had a height of 0.095 in. which was machined down for other geometries. Dimensions of the specimens are given in Fig. 4. The specimens were clamped in aluminum end plates

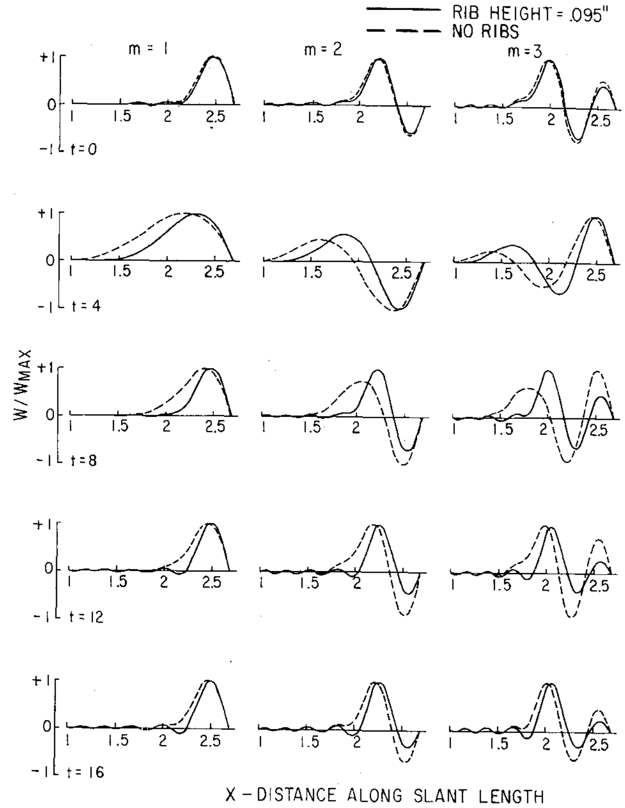

Fig. 8 Comparison of modal shapes.

containing a trough filled with cerrobend, a low-melting-point alloy.

Results and Discussion

The effect on the natural frequency of increasing the ratio E_ϕ/E_x but keeping E_x and $\nu_{\phi x}$ constant was investigated. This was done for a particular conical shell with the geometry given in Table 1. Calculations were made on an IBM 7090 digital computer using Eqs. (10).

Four elastic constants are needed as input for Eqs. (10) to determine the frequency spectra. These were accounted for in the following manner: 1) two were input values $(E_\phi/E_x, \nu_{\phi x})$, 2) the frequency parameter $\bar{\Omega}$ was determined in terms of the elastic constant E_x, and 3) the remaining elastic constant was computed by the approximate equation given by Marcus[30]:

$$E_x/G = [1 + \nu_{\varphi x}(E_x/E_\varphi)] + (E_x/E_\varphi)(1 + \nu_{\varphi x})$$

Typical computed results of the frequency parameter for the first three modes are tabulated as a function of the circumferential wave number t in Table 1 (see Ref. 31 for the complete set of results). In addition, the frequency parameter for an equivalent cylindrical shell is also tabulated in Table 1. These values were obtained by using the equivalent cylinder geometry relations given in Ref. 31 in conjunction with Eqs. (10) for the semivertex angle going to zero. The minimum frequency prediction of the equivalent cylindrical shell for all E_ϕ/E_x values are in good agreement with the present theory predictions. The equivalent cylinder predictions at low wave numbers are in better agreement with the present theory than at high wave numbers. Minimum values for the frequency parameter are plotted as a function of E_ϕ/E_x in Fig. 4. The slope of the curve yields the relationship $f_{\min}\alpha(E_\phi/E_x)^{1/4}$ for the particular geometry investigated. This indicates that only very large changes in E_ϕ/E_x will effect the minimum frequency parameter.

An experimental investigation was carried out for both a ring-stiffened conical frustum and a ring-stiffened cylindrical shell. The ribs were machined such that the assumptions for reducing a ring-stiffened conical shell to an equivalent orthotropic shell would apply. The geometry of the specimens are given in Fig. 5. The experimental results and corresponding theoretical predictions obtained from Eqs. (10–23) are shown graphically in Figs. 6 and 7.

In general, the experimental results follow the theoretical predictions with a maximum deviation of 17% for the cylindrical shell and 6% for the conical shell. This variance occurred for a rib height of 0.095 in. and a large circumferential wave number. It should be noted from Figs. 6 and 7 that the difference in the minimum frequency for the various rib heights at a given mode number is only slight. The frequency curves separate from one another as the wave number increases. This agrees with the behavior of the orthotropic shell at the minimum frequencies.

A possible physical explanation for the small difference of the minimum frequencies is that, as the wave number increases, the ring stiffeners are bent into shorter wavelengths. This makes the stiffness of the shell greater, which results in higher natural frequencies for shells with rib stiffeners. A plot of the modal shapes as a function of circumferential wave number is given in Fig. 8. The mode shapes indicate that the axial wavelength decreases with increasing wave number. The axial wavelength for certain modes is also smaller for the rib-stiffened shell, which probably partially accounts for the higher frequency of the stiffened shell.

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